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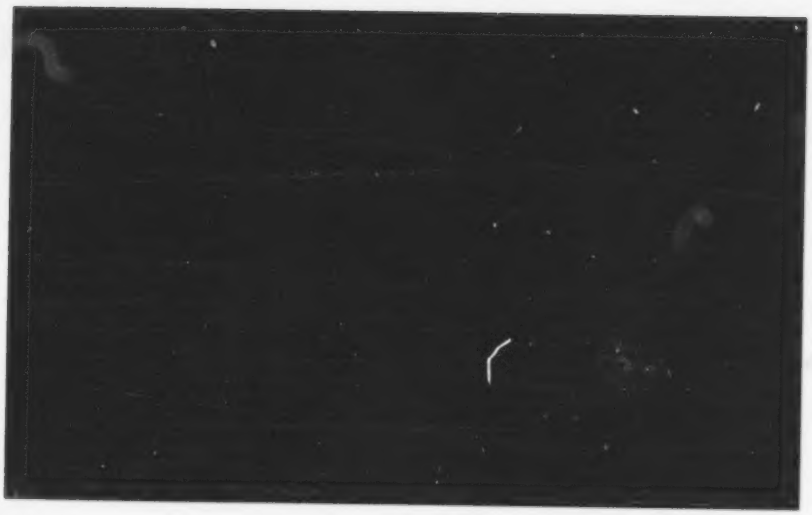

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A VARIATIONAL INEQUALITY FORMULATION OF THE DYNAMIC TRAFFIC ASSIGNMENT PROBLEM

by

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Abstract

This paper is concerned with the modelling of the Dynamic Traffic Assignment Problem (DTAP) for analysing the day time varying flows of urban transportation networks. During the past two decades, many models have been proposed in the literature, but some of them are based on heuristic concepts, and most of them incorporate important limitations.

In this paper, we propose a Dynamic Traffic Assignment Model which is mainly based on the following assumption: the time spent by a vehicle on a link may be decomposed into a fixed travel time and a waiting time. The fixed travel time is the free flow travel time, after which vehicles are put in the link exit queue, until that there is room for them to proceed their trip. We show that this model leads to a network structure (a temporal expansion of the base network, including the queues), then, we formulate the DTAP as a network equilibrium problem over the expanded network. The mathematical formulation is achieved through a variational inequality where the variables are the path (or link) flows over the space-time expanded network. Numerical results show that the model may handle large networks, so it may be used in practice to analyse the traffic congestion moves in real cities, as well in the space (physical links) as in the day time.

Key words: Dynamic, Traffic, Assignment, Network, Equilibrium, Variational inequality.

Résumé

Cet article concerne le problème d'affectation dynamique du trafic pour l'analyse des réseaux de transport urbain aux heures de pointes. Les études effectuées dans le passé ont abouti soit à des heuristiques, soit à des modèles d'optimisation comportants des limitations importantes.

Dans cet article, nous proposons un modèle d'affectation dynamique du trafic, principalement basé sur les hypothèses suivantes: le temps passé par un véhicule sur un lien peut être décomposé en un temps de traversée fixe plus un temps d'attente. On montre que ce modèle dynamique peut être perçu comme un modèle d'affectation statique sur une expansion temporelle du réseau. Le problème est alors formulé, à l'aide d'une inégalité variationnelle, comme un problème d'équilibre de flot sur un réseau (espace-temps). Les résultats numériques montrent que ce modèle peut servir en pratique pour analyser le déplacement de la congestion de trafic urbain, dans l'espace et dans le temps.

Mots clés: Affectation, Dynamique, Trafic, Réseau, Équilibre, Inégalité variationnelle.

1- INTRODUCTION

This paper is concerned with the modelling of the traffic flows on urban transportation networks, mainly during the peak periods of a day. In the past, most of the research has been concentrated in the Static Traffic Assignment Problem (STAP) and its variants, where a *static* demand is assigned on the network under the basis of the descriptive assignment rules of Wardrop [1952] (see, among many others, Dafermos and Sparrow [1969]). Altought the demand is not constant over the day time in urban contexts, the static models have been extensively used in practice.

During the two past decades, research has been more and more concentrated in developping new tools regarding the Dynamic Traffic Assignment Problem (DTAP), that may help to analyze the traffic flow variations over the time, in a city transportation network. The early DTAP models were often based on heuristic concepts (Yagar, [1970], Leonard et al., [1978]) where the traffic flows are assigned on *instantaneous* minimum cost paths. Other models were too restrictive: some do not apply to capacity constrained links (Robillard, [1974]) and others consider only simple networks with only one destination (Hendrickson and Kocur [1981]) or having only parallel paths between the (only) origin-destination pair (Arnott et al., [1987]). Finally, some impose some simplifying conditions on the users' knowledge of the traffic conditions over the time (Drissi, [1990]).

In 1978, Merchant and Nemhausser present a first (discret time nonconvex) minimization model for the single origin-destination DTAP. Ho [1980] proposes a stepwise linear version of this model which has been more deeply investigated by Carey [1986 and 1987]; this author presents a convex version of the model and some techniques to generalize it to the many origins and many destinations case. Finally, Friesz et al. [1989] propose a continuous time version of the model (optimal control problem). These models are based on the following representation of the links' dynamic of flow:

$$x_a^{t+1} = x_a^t + d_a^t - g_a(x_a^t) \;\; \forall a \in A, t = 0, 1, .., T-1$$

where:

x_a^t and x_a^{t+1} are the flows on link a at periods t and $t+1$,
d_a^t is the flow that enters link a at period t,
$g_a(x_a^t)$ is the flow that exits link a at period t,
A is the set of links of the network, and
T is the number of periods of study (horizon).

In this representation of the link flows dynamic, the flow that exits a link a is given by mean of an "exit function" $g_a(.)$. In order that their formulation be convex, the exit functions are restricted to be concave, which is not necessarily satisfied in congested networks. Further, the dynamic flow conservation constraints of the model do not ensure that the flow will reach the destination (independently of the horizon T); in order to overcome this problem, the authors restricted the link cost functions to satisfy a condition that is, roughly: the cost of a link must increase with its distance to the destination. Finally the

induced mathematical program have not any special structure that may lead to a particularly efficient adaptation of mathematical programming algorithms for the solution of the problem.

In the past, Ford and Fulkerson [1962] and Fulkerson [1975] proposed a model for the Dynamic Maximal Flow Problem, where each link $a \in A$ has a capacity and a fixed traversal time, and for any period of time, at each node vehicles can either be trans-shipped on exiting links or wait until the next period. The resulting space-time network is built as follows:

- each node $n \in N$ is expanded to $T+1$ nodes n_t, $t = 0, 1, .., T$, and
- each link $a = (i, j) \in A$ is expanded to the links $(i_t, i_{t+\tau_a})$ where τ_a is the fixed traversal time on link a and t such that $t \geq 0$ and $t + \tau_a \leq T$; we refer to these links as travel links.
- the waiting time at the nodes is represented by mean of the links (n_t, n_{t+1}) where $n \in N$ and $t = 0, 1, .., T-1$; we refer to these links as the queues.

Figure 2 illustrates an expansion of the network of figure 1, according to the model described above.

Figure 1: A simple network

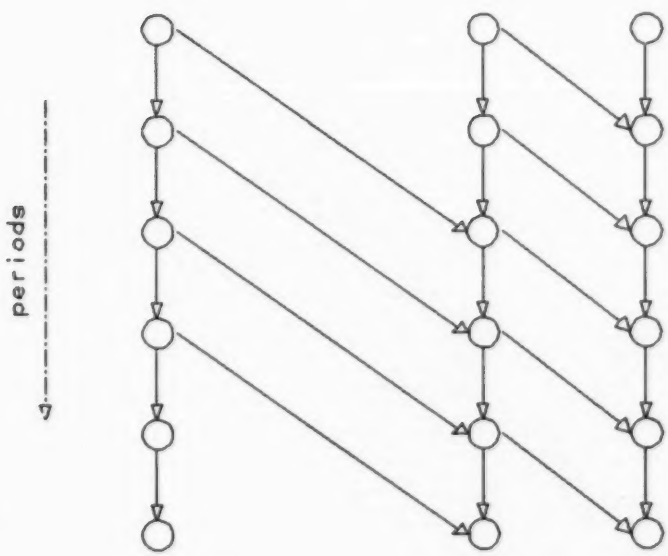

Figure 2: Expansion of the simple network

While this model was not specificly designed for the analysis of urban transportation networks, it predicts that when there is no congestion on the network, the time spent on each link is constant (free or uncongested traversal time), and, in presence of congestion, it increases because of waiting in queues; this meets our expectation. However, in this model, the queues hold at the nodes rather than on the links, and no capacity is associated with the queues volumes, which would mean that the nodes of the city network (intersections) would be viewed as infinite stacks for vehicles.

Recently, Zawack and Thompson [1987] adapted this model for the DTAP with the terminology of green links for the travel links and red links for the queues. They formulated the DTAP as a trans-shippement problem over their expanded network, but they consider only one destination and many origins. While they brought some improvements to the space-time expanded network of Ford and Fulkerson [1962] and Fulkerson [1975] (finite capacities are set on the queues, and some unnecessary nodes and links of the expanded network are identified), the problem of queues at the nodes has not been completely resorbed by these authors. The fact is that it is not clear how it is possible to characterize adequately the node queues capacities since the queues are physically on the links (streets) rather than on the nodes (intersections), and the queue on a link shares the capacity of the link with the other vehicles on the same link (but not yet in the queue). Setting the queues at the nodes, as in the models above, means that the queues of all the links sharing a same head node are grouped together to form a single node queue. But each of the components (link queues) of a node queue has its own capacity that it does not necessarily share with the other components. As a consequence, it seems that it is not possible to define global fixed node queues capacities, as suggested by Zawack and Thompson [1987]. Our point of view is that a more realistic model must rather incorporate link queues as suggested in Drissi and Hameda [1990].

The remainder of this paper is organized as follows: in the two next sections, we describe a space-time dynamic network in the spirit of those of Ford and Fulkerson [1962], Fulkerson [1975] and Zawack and Thompson [1987], where we incorporate explicit link queues, and discuss the link costs and capacities. In section 4, we state the equilibrium conditions and formulate the Dynamic Traffic Assignment Problem as a mutli-commodity (many origins and many destinations) network equilibrium problem, by mean of a variational inequality. We propose a solution algorithm in section 5, and give some numerical results in section 6.

2. THE NETWORK TEMPORAL EXPANSION

In this section, we state our assumptions on the flows dynamic on the network, and then we show that they lead to a space-time network structure.

2.1- Assumptions

The main assumptions underlying our model are the following:

-A1- For any period of time the (nonnegative) link flows are bounded (from above) by the link capacities; hence, a vehicle cannot enter a congested link (e.g. at full capacity); it must wait (on the link in which it is) until that there is room for it to proceed its trip.

-A2- The total time spent by a vehicle on a link a may be decomposed into two parts that are:

- the fixed travel time τ_a which is the free flow (uncongested) travel time on link a, after which the vehicle is ready to exit the link (e.g. the vehicle joins the exit queue of the link), and

- the waiting time which is the time spent in the exit queue of the link; according to A1, a vehicle has to wait if the link in which it must enter is congested (e.g. at full capacity).

-A3- The free travel time τ_a of link a is measured in numbers of periods of time.

2.2- The Static Temporal Expanded Network (STEN)

According to the assumptions above, a vehicle that enters a link a at period t, must travel during τ_a periods of time, and then it has the "choice"[1] of either exiting the link or joining the exit queue of the link at period $t + \tau_a$; in this case, after one period of time, it will have again a choice of exiting the link or joining the exit queue of the link corresponding to the time period $t + \tau_a + 1$, and the process is repeated until that the vehicle exits the link. This process may be described as a path following in the space-time network that we construct in what follows:

• Nodes: each node n_k, $(k \in N)$ of the base network is expanded to $(T+1)$ nodes n_k^t, $t = 0, 1, .., T$, $(k \in N)$ of the expanded network.

• Links: for each link $a = (n_i, n_j)$, $\in A$, we construct, for each $t = 0, 1, .., T$:
- one node l_a^t,
- one link $(n_i^{t-\tau_a}, l_a^t)$ if $t - \tau_a \geq 0$,
- one link (l_a^t, n_j^t) and
- one link (l_a^t, l_a^{t+1}) if $t + 1 \leq T$.

These three links correspond respectively to the fixed travel time on the link, the exit part of the process and the queue at the given time period t.

In order not to allow more vehicles than possible to enter the network at the origin nodes, the same approach of waiting queues is used at the origin nodes:

• Origin nodes: for each origin node n_k, $k \in O$, we construct, for each $t = 0, 1, .., T$:
- one node $L_{n_k^t}$,
- one link $(L_{n_k^t}, n_k^t)$ and
- one link $(L_{n_k^t}, L_{n_k^{t+1}})$ if $t + 1 \leq T$.

[1] This choice is based on some criteria that will be discussed later in this paper.

These two links correspond respectively to the entry of flow into the network and the waiting queue a the origin node n_k during the time period t.

- Destination nodes: for each destination node n_k, $k \in D$, we construct:
  - one node d_k and
  - one link (n_k^t, d_k) for each $t = 0, 1, .., T$.

This link corresponds to the exit of flows of the network at destination node n_k during period t.

In order to illustrate this network expansion process, we consider the simple network of figure 1, and show the corresponding static expanded network (for five periods) in figure 3; we consider that n_1 is an origin node, n_3 a destination node and n_2 a regular node.

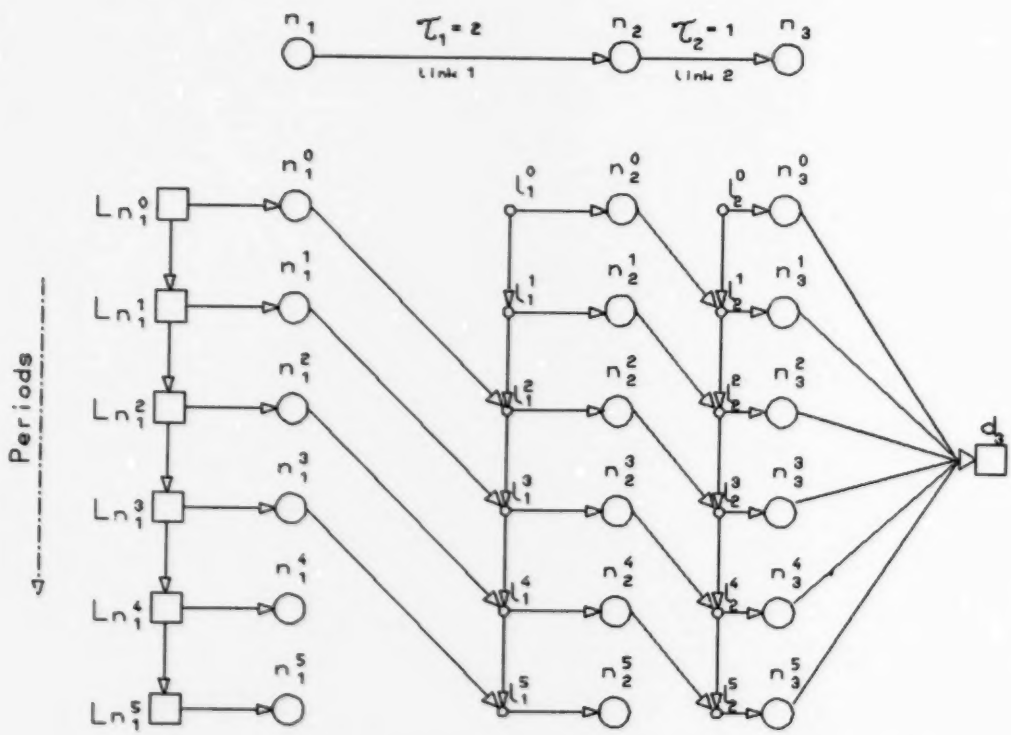

Figure 3: Expansion of the simple network of Fig. 1

In the next subsection, we describe the dynamic of the transfer of flows within and between the links of the original network by mean of those of the STEN.

2.3- Flows on the STEN

In this section, we show that the dynamic flows on the original network, may be seen as static flows on the STEN.

Let A be the set of links of the original network, and T the horizon of time (number of periods). Then, let's define, for each $a \in A$ and each time period t:

u_a^t : the flow that enters in link a during period t,
v_a^t : the flow that exits from link a during period t,
q_a^t : the flow that waits on link a during period t,

The relation that expresses the dynamic process of transfer of flow *within* a link of the original network, is the simple static flow conservation constraint at the nodes l_a^t, $a \in A$, $t = 0,..,T$, of the STEN.

$$q_a^t = q_a^{t-1} + u_a^{t-\tau_a} - v_a^t \quad \forall a \in A, \quad t = 0,..,T \tag{1}$$

The transfer of flow *between* the links (e.g. at the nodes) is such that the arriving flow to a node at a given time period, is equal to the exiting flow from that node at the same time period; this is expressed by the static flow conservation constraint at the nodes n_k^t of the STEN:

$$b_k^t + \sum_{a \in B_k} v_a^t = \sum_{a \in A_k} u_a^t + e_k^t \quad \forall k \in N, \quad t = 0,..,T \tag{2}$$

where

B_k is the set of links arriving to the node k,
A_k is the set of links exiting from the node k,
b_k^t is the entry flow in the network at node n_k^t (zero if k is not an origin), and
e_k^t is the exit flow of the network at node n_k^t (zero if k is not a destination).

Let g_k^t be the new input at origin node n_k, $k \in O$ during period t, and Q_k^t the origin node queue. The entry flow dynamic during period t at the origin node n_k is expressed by the flow conservation at the node $L_{n_k^t}$ of the STEN:

$$Q_k^t = Q_k^{t-1} + g_k^t - b_k^t \quad \forall k \in O, t = 0,..,T \tag{3}$$

The last equations are relative to the destination nodes that are d_k in the STEN:

$$\sum_{t=0}^{T} e_k^t = E_k \quad \forall k \in N \tag{4}$$

where E_k is the total demand at destination node k.

Therefore, all the dynamic flow moves in the original network, may be seen as static flow moves in the STEN. In the following, we address the links' costs and capacities.

3- LINKS PERFORMANCE

The previous section dealed with the STEN topology. In order to complete its characterization, it remains to describe the links costs and capacities. We discuss two approaches where the links capacities are expressed either explicitly or implicitly.

3.1- Links explicit capacities and travel costs

Each link of the STEN has a capacity that is an upper bound on its flow. The capacities relative to a link entry, exit or queueing vehicles, are not necessarily the same; therefore, for each link $a \in A$, and for each period $t = 0,..,T$, we set:

$$\text{- Link entry capacity} \quad : \quad u_a^t \leq C_a^u \tag{5}$$

$$\text{- Link exit capacity} \quad : \quad v_a^t \leq C_a^v(v) \tag{6}$$

$$\text{- Link queue capacity} \quad : \quad q_a^t \leq C_a^q \tag{7}$$

The entry and queue capacities of a link may be taken to be constants. We may even accept the assumption that $C_a^q = \tau_a C_a^u$, that is the maximum flow accumulated during τ_a periods on link a. As for the exit capacity of a link, it may also be taken to be constant for simplicity, but in general, the exit flow capacity depends on the traffic conditions of other links at the same intersection, as well as on the geometric and signalization conditions at the intersection; in this case, the exit capacity of a link must not be constant, but rather a function of the vector of exit flows v (at least v^t that are the exit flows at the current period of time). More details on such functions may be found in the Highway Capacity Manual, 1985.

As for the transportation costs on these links, it seems appropriate to relate them to the travel times that are:

$$\text{- for a travel link } u_a^t \quad : \quad \tau_a \quad \text{periods} \tag{8}$$

$$\text{- for an exit link } v_a^t \quad : \quad 0 \quad \text{period} \tag{9}$$

$$\text{- for a waiting link } q_a^t \quad : \quad 1 \quad \text{period} \tag{10}$$

The travel costs may then be these travel times multiplied by some appropriate time-money conversion factors.

For the links at the entry and exit of the network (e.g. origins and destinations), the capacities may be taken to be infinite, and the costs zero for the b_k^t and e_k^t links, and one for the queues Q_k^t.

3.2- Links implicit capacities and travel costs

The classical strategy adopted in the literature of static equilibrium problems, is to combine to the uncongested link costs, the effect of congestion (capacities) as a nonlinear component in the total link cost. The same approach may be used here for the links of the STEN. Therefore, we may combine the links costs and capacities discussed in the previous subsection, into single nonlinear link costs:

$$\text{- for a travel link } u_a^t \quad : \quad \alpha\tau_a + \quad \mu(\max(0, u_a^t - C_a^u))^{\delta_a^u} \tag{11}$$

$$\text{- for an exit link } v_a^t \quad : \quad \mu(\max(0, q_a^t - C_a^v(v)))^{\delta_a^v} \tag{12}$$

$$\text{- for a waiting link } q_a^t \quad : \quad \beta + \quad \mu(\max(0, q_a^t - C_a^q))^{\delta_a^q} \tag{13}$$

where α and β are (eventually different) time-money conversion factors for travel and waiting times, and μ is a large constant so that excess flows on the links would be penalized; δ usually lies between 2 and 5. Of course, other functional forms for combining the uncongested costs and capacities, that could better describe some particular practical situations, are welcome. In the remainder of this paper, we adopt this strategy of implicitly handling the capacities through nonlinear costs.

4- THE MATHEMATICAL MODEL

In this section, we consider the STEN $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ characterized in sections 2 and 3. Let $g_{n_p^t d_q}$ be the demand during period t at the origin node n_p for the destination node n_q (d_q in the expanded network); $g_{n_p^t d_q}$ must be assigned on the (space-time) paths $k \in K_{pq}^t$, where K_{pq}^t is the set of paths between the nodes n_p^t and d_q in the STEN $\mathcal{G}$.

In order to construct a mathematical model that performs the assignment of the demand on the paths of $\mathcal{G}$, we still need to make some assumptions on the users' behaviour and their knowledge of the system. These are the following:

-UA1- Each user strives to minimize its own travel cost (selfish assignment principle).

-UA2- Each user knows the topological characteristics of the network.

-UA3 Users have a long term knowledge of the traffic conditions.

In other words, we assume that the users know the traffic conditions on the network over all the study horizon of time. In practice, this knowledge is acquired by experience, so that users know the impact of their route choices on the traffic conditions of the whole system.

In the following, we consider the expanded network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, and we need no more distinguish between the different types of links of $\mathcal{A}$ (e.g. if they correspond to a travel, exit or queue link). This network is self consistent: it has a set of origin nodes $\{L_{n_k^t}, k \in$

$O, t = 0, .., T\}$, a set of destination nodes $\{d_k, k \in D\}$, (space-time) paths between the origin and destination nodes, and with each link of $\mathcal{A}$, is associated a (nonlinear) travel cost function that expresses the travel time component as well as the link capacity (see for instance section 3.2).

A (space-time) path cost is defined to be the sum of the costs of the links[2] that belong to that path. Therefore, the cost $S_k(h)$ of a path k in $\mathcal{G}$ is given in term of the costs $s_a(f)$ of the links $a \in \mathcal{A}$, by:

$$S_k(h) = \sum_{a \in \mathcal{A}} \delta_{ak} s_a(f) \tag{14}$$

where:

$$\delta_{ak} = \begin{cases} 1 & \text{if link } a \text{ is in path } k \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

In equation (14), f and h refer respectively to the vectors of link flows $f_a, a \in \mathcal{A}$, and path flows $h_k, p \in O, q \in D, t = 0, .., T, k \in K_{pq}^t$. The relation between the link and path flows is that the flow on a given link is the sum of all those of the paths passing on this link; this is expressed formally by:

$$f_a = \sum_{p \in O} \sum_{q \in D} \sum_{t=0}^{T} \sum_{k \in K_{pq}^t} \delta_{ak} h_k \quad \forall a \in \mathcal{A} \tag{16}$$

The assignment of the demand on the paths of the expanded network $\mathcal{G}$ must be done in such a way to satisfy the path flow conservation and nonnegativity constraints:

$$\sum_{k \in K_{pq}^t} h_k = g_{n_p^t d_q} \quad \forall p \in O, \ \forall q \in D, \ t = 0, 1, .., T \tag{17}$$

$$h_k \geq 0 \quad \forall p \in O, \ \forall q \in D, \ t = 0, 1, .., T, \ \forall k \in K_{pq}^t \tag{18}$$

and, according to our assumptions, in a way to satisfy the selfish assignment principle:

$$S_k(h) \begin{cases} = & u_{pq}^t \quad \text{if} \quad h_k > 0 \\ \geq & u_{pq}^t \quad \text{if} \quad h_k = 0 \end{cases} \quad \forall p \in O, \forall q \in D, t = 0, .., T, \forall k \in K_{pq}^t \tag{19}$$

where

$$u_{pq}^t = \min_{k \in K_{pq}^t} \{S_k(h)\} \tag{20}$$

is the cost of the shortest path from origin node p at period t, to the destination node q, given the flows h on the network $\mathcal{G}$.

These assignment conditions are known to be the equilibrium conditions or descriptive assignment rules of Wardrop (1959); they state that all the paths used between an origin at a given time and a destination, must have the same cost that is smaller (or equal) than the

[2]These are links of the expanded network, including travel and exit links, as well as waiting links (queues).

costs of the unused paths between the same origin node, time period and destination node. These conditions reflect the selfish users' behaviour UA1, since if a user tries to change individually its route, then it could only increase its own travel cost.

Theorem 1: Let Ω_h be the set of feasible path flows defined by equations (17) and (18), then, the equilibrium conditions (19) may be rewritten as a variational inequality:

$$\text{Find } h \in \Omega_h \text{ such that } \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t}(S_k(h)-u_{pq}^t)(h_k'-h_k)\geq 0 \quad \forall h'\in\Omega_h \tag{21}$$

Proof: Since the path flows are nonnegative, then, for any $p \in O, q \in D, t = 0,..,T, k \in K_{pq}^t$, two cases may happen:

. $h_k = 0$, in which case $(S_k(h) - u_{pq}^t)(h_k' - 0) \geq 0$ by (19) and the fact that $h_k' \geq 0$.

. $h_k > 0$, in which case $(S_k(h) - u_{pq}^t)(h_k' - h_k) = 0$ by (19),

then summing over $p \in O, q \in D, t = 0,..,T, k \in K_{pq}^t$, leads to (21).

Conversely, assume that (21) is satisfied, then, in particular, the inequality must hold for $\bar{h} \in \Omega_h$ defined as follows:

$$\begin{cases} \bar{h}_{k_{pq}^t} = g_{pq}^t & \text{for a path } k_{pq}^t \text{ such that } S_{k_{pq}^t}(h) = u_{pq}^t \\ \bar{h}_k = 0 & \forall k \in K_{pq}^t, k \neq k_{pq}^t \end{cases} \quad \forall p \in O, \forall q \in D, t = 0,..,T$$

Then

$$\begin{aligned}\sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t}(S_k(h)-u_{pq}^t)(\bar{h}_k-h_k) &= \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t,k\neq k_{pq}^t}(S_k(h)-u_{pq}^t)(0-h_k)\\ &\quad+\sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}(u_{pq}^t-u_{pq}^t)(g_{pq}^t-h_{k_{pq}^t})\\ &= \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t,k\neq k_{pq}^t}(S_k(h)-u_{pq}^t)(-h_k)+0\end{aligned}$$

But since $(S_k(h) - u_{pq}^t) \geq 0$ and $h_k \geq 0$, then $(S_k(h) - u_{pq}^t)(-h_k) \leq 0 \; \forall p \in O, q \in D, t = 0,..,T, k \in K_{pq}^t$. Therefore, $\sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t}(S_k(h) - u_{pq}^t)(-h_k) \geq 0$ if and only if $h_k > 0$ implies that $(S_k(h) - u_{pq}^t) = 0 \; \forall p \in O, \forall q \in D, t = 0,..,T, \forall k \in K_{pq}^t$. •

The variational inequality (21) is equivalent to the following one (22), that we call our *path flow variational inequality formulation* of the dynamic traffic assignment problem:

Theorem 2: Variational inequality (21) is equivalent to variational inequality (22):

$$\text{Find } h \in \Omega_h \text{ such that } \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t}S_k(h)(h_k'-h_k)\geq 0 \quad \forall h'\in\Omega_h \tag{22}$$

Proof: Since, in (21), $h \in \Omega_h$ and $h' \in \Omega_h$, they satisfy the flow conservation constraint (17); therefore, it follows that:

$$\sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t}(S_k(h)-u_{pq}^t)(h_k'-h_k) =$$

$$= \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t} S_k(h)(h_k'-h_k) - \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t} u_{pq}^t(h)(h_k'-h_k)$$

$$= \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t} S_k(h)(h_k'-h_k) - \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T} u_{pq}^t \sum_{k\in K_{pq}^t}(h_k'-h_k)$$

$$= \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t} S_k(h)(h_k'-h_k) - \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T} u_{pq}^t(g_{pq}^t-g_{pq}^t)$$

$$= \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t} S_k(h)(h_k'-h_k) - 0\bullet$$

Theorem 3: The problem has also a *link flow variational inequality formulation* that is:

$$\text{Find } f\in\Omega_f \text{ such that } \sum_{a\in\mathcal{A}} s_a(f)(f_a'-f_a)\geq 0 \quad \forall f'\in\Omega_f \tag{23}$$

where Ω_f is the set of feasible link flows over $\mathcal{G}$, defined by equations (15) to (18).

Proof: If we use equations (14) and (16) to rewrite the left side term of (22) in term of link flows, then:

$$\sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t} S_k(h)(h_k'-h_k) = \sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t}\sum_{a\in\mathcal{A}}\delta_{ak}s_a(f)(h_k'-h_k)$$

$$= \sum_{a\in\mathcal{A}} s_a(f)\sum_{p\in O}\sum_{q\in D}\sum_{t=0}^{T}\sum_{k\in K_{pq}^t}\delta_{ak}(h_k'-h_k)$$

$$= \sum_{a\in\mathcal{A}} s_a(f)(f_a'-f_a)\bullet$$

From the classical theory of variational inequalities, it is known that the *link flows* equilibrium solution of (23) is unique if the cost functions are strictly monotone, e.g., $\sum_{a\in\mathcal{A}}(s_a(f')-s_a(f''))(f'-f'')>0, \forall f'\in\Omega_f, \forall f''\in\Omega_f$. However, since the link flows have not necessarily a unique path decomposition, then the *path flows* equilibrium solution is not necesary unique.

Finally, from a classical result of the theory of numerical analysis, known as the symetry principle (Ortega and Rheinboldt, 1970), if the jacobian matrix $\nabla s(f)$ is symetric, then the variational inequality above, has an equivalent optimisation problem. In the particular case where the link costs are seperable, that is: $s_a(f)=s_a(f_a), a\in\mathcal{A}$, the above variational inequality formulation is equivalent to the following mathematical programming problem:

$$\min \sum_{a\in\mathcal{A}}\int_0^{f_a} s_a(z)dz \text{ s.t. } f\in\Omega_f \tag{24}$$

For instance, if the exit capacities are set to be constant $C_a^v(v) = C_a^v$, then, according to equations (12), (13) and (14) the link costs are seperable, and in this case, the problem formulation is equivalent to (24).

5- SOLUTION OF THE PROBLEM

In this section, we are concerned with the solution of the link variational inequality formulation (23) of the dynamic traffic assignment problem.

The solution algorithm that we suggest is the diagonalization or Jacobi method, which consists in solving a sequence of variational inequalities VI^l, $l = 1, 2, ..$, defined as follows:

$$VI^l : \text{Find } f^{l+1} \in \Omega_f \text{ such that } \sum_{a \in \mathcal{A}} s_a^l(f_a^{l+1})(f_a' - f_a^{l+1}) \geq 0 \quad \forall f' \in \Omega_f \qquad (25)$$

where : $s_a^l(f_a) = s_a(f_1^l, ..., f_{a-1}^l, f_a, f_{a+1}^l, ..., f_{|\mathcal{A}|}^l)$.

Florian and Spiess (1982) have shown that this method converges to the equilibrium solution if the jacobian matrix $\nabla s(f)$ of the link cost functions, is diagonal dominant.

Obviously, each subproblem VI^l, $l = 1, 2, ..$, is separable, so it has an equivalent optimization problem of the form of (24). Therefore, the solution algorithm reduces to the solution of a sequence of minimization problems such (24); hence, in the remainder of this section, we concentrate on the solution of problem (24).

Note that Ω_f is a polytope (linear constraints), which is nonempty[3], and compact as well if the cost functions on all the links of $\mathcal{A}$ are strictly positive (Guélat et al. [1990])[4]. Hence, in order to solve (24), we suggest to use the most popular algorithm for solving large scale STAP's, which is the Linear Approximation algorithm (Frank and Wolfe [1956]) that reduces to a sequence of minimum cost paths computations and 1-directional minimizations (see Guélat et al. [1990] for instance). This algorithm may be stated as follows:

[3] We assume that there exist at least one path from each origin to each destination in $\mathcal{G}$, on which all the demand may be assigned.

[4] The paths considered do not contain cycles.

Linear Approximation Algorithm

Step 0 Determine initial feasible solution $f \in \Omega_f$

Step 1 1.1- Determine extremal point w
1.2- Compute descent direction $d = w - f$
1.3- Compute optimal step size λ
1.4- Update flows $f = f + \lambda d$

Step 2 If $\|\lambda d\| < \epsilon$ then stop
otherwise return to step 1.

In step 1.1 of the algorithm, the extremal point w is the optimal solution of the linear approximation of (24) at the current solution f:

$$\min \sum_{a \in \mathcal{A}} \int_0^{f_a} s_a(x)dx + s_a(f_a)(w_a - f_a) \quad s.t. \quad w \in \Omega_f \tag{26}$$

which is equivalent to:

$$\min \sum_{a \in \mathcal{A}} s_a(f_a) w_a \quad s.t. \quad w \in \Omega_f \tag{27}$$

but $w_a = \sum_{p \in O} \sum_{q \in D} \sum_{t=0}^{T} \sum_{k \in K_{pq}^t} \delta_{ak} y_k \quad \forall a \in \mathcal{A}$, where y is the vector of path flows corresponding to w. Hence, (27) may be rewritten:

$$\min \sum_{a \in \mathcal{A}} s_a(f_a) \sum_{p \in O} \sum_{q \in D} \sum_{t=0}^{T} \sum_{k \in K_{pq}^t} \delta_{ak} y_k \quad s.t \quad y \in \Omega_h \tag{28}$$

or

$$\min \sum_{p \in O} \sum_{q \in D} \sum_{t=0}^{T} \sum_{k \in K_{pq}^t} y_k \sum_{a \in \mathcal{A}} \delta_{ak} s_a(f_a) \quad s.t \quad y \in \Omega_h \tag{29}$$

or, by using equation (14), ,

$$\min \sum_{p \in O} \sum_{q \in D} \sum_{t=0}^{T} \sum_{k \in K_{pq}^t} y_k S_k(h) \quad s.t. \quad y \in \Omega_h \tag{30}$$

Hence, the optimal solution is clearly obtained by an all or nothing assignment of the demand on the minimum cost paths; the corresponding extremal point in terms of link flows w is obtained by:

$$w_a = \sum_{p \in O} \sum_{q \in D} \sum_{t=0}^{T} \sum_{k \in K_{pq}^t} \delta_{ak} y_k \quad \forall a \in \mathcal{A}$$

6- NUMERICAL RESULTS

In this section, we present some preliminary numerical results that show interesting properties of the model.

The test network that we used is a real transportation network (network of the city of Hull, Canada) that has about 800 links, 500 nodes and 23 centroids. The study horizon of time was splitted into 30 periods of time. Therefore, the corresponding the expanded network has about 74000 links and 40000 nodes.

An important property of the model, that should be mentionned here, is that the topology of the expanded network is fully described by the information on the topology of the original network and the expansion procedure described in section 2.2. Therefore, in the computer implementation of the solution algorithm, it has not been necessary to store explicitly the topology of the expanded network; this has led to an efficient adaptation of Dikjstra's (1959) label setting method for computing the trees of shortest paths required to solve each subproblem of the linear approximation algorithm. This is certainly one of the most important factors that make large (real) networks tractable by this model. The code has been written in Fortran 77 on the SUN 3/50 (MC68020/15Mhz) at the Centre de recherche sur les transports, Université de Montréal. Each iteration of the linear approximation algorithm took about 10 minutes, and after 10 iterations, the gap[5] was less than 1%. Approximatively, half of the computation time was used in the shortest path subroutine and most of the other half in the line search subroutine. The computation time performance on the network of Hull, Canada, then shows that the model may be used to analyse the flow variations of real city networks.

We have compared the *total* flows on the links (e.g. $\sum_{t=0}^{T} v_a^t,\ a \in A$) with the flows on the links obtained by the classical static traffic assignment model. While we met our expectation by obtaining a good fit between the two sets of flows, it remains to do more intensive numerical tests for comparing the link flows and queues for the different periods of time, with observed ones; we are seeking the city of Montréal to provide us some recent data (link flow counts for differents periods of time) that will help us achieve this purpose.

[5] gap = 100.(obj - blb)/obj, where obj is the value of the objective function, and blb the best lower bound obtained for the objective function.

7- CONCLUSION

This paper is concerned with the modelling of the Dynamic Traffic Assignment Problem (DTAP) for predicting the flows of urban transportation networks during the peak periods.

During the past twenty years, many models have been proposed in the literature, but all of them incorporate important limitations; many of them are based on heuristic concepts, others are restrictive and consider only the case of parallel paths between one origin-destination pair or only one destination.

Recently, a model that was proposed by Ford and Fulkerson [1962] and Fulkerson [1975] for the dynamic maximal flow problem, has been adapted by Zawack and Thompson [1987] to the DTAP; this model is based on the assumption that the time spent by a vehicle on a link may be separated in two parts:

- a fixed traversal time, and
- a waiting time at the head node.

The fixed traversal times refer to the travel times when no congestion is present, and the waiting times are related to the delays induced by finite capacities on the links during congestion periods. This model leads to a static assignment model over a temporal expansion of the base network that may include many origins, but only one destination.

The model that we propose in this paper is also based on the assumption that the time spent by vehicles on the links may be decomposed into a fixed travel time plus a waiting time in an exiting queue, but, in our model, the queues are set on the links so that the vehicles in the queue of a link share the physical space with the other vehicles on the link. Hence, we express the model as a Static Traffic Assignment Problem (over a temporal expansion of the base network). The mathematical formulation of this model is a variational inequality where the variables are the feasible path (or link) flows over the space-time expanded network.

The numerical results that we obtained show that this model may be used to simulate the flows of real large networks. The model provides to the planner, among other important output, the flows and queues on the network for each period of time are produced, which helps to analyse the moves of congestion in the space (physical links) and in the time.

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